4. GEOMETRY

§4.1. Euclid's Geometry

Basic mathematics consists of arithmetic, algebra and geometry. Of these, by far the most interesting is geometry. A picture is worth a thousand symbols. We are much more at home visualising something than describing it algebraically.

Before Euclid mathematics was an experimental science. When the pyramids were built it was known that a 3-4-5 triangle gives you a right angle, and the four sides of a square pyramid have to include a right-angle.

Since the Babylonians angles have been measured in degrees and they decided to make 360 degrees a full revolution, which makes a right-angle 90 degrees. The Babylonians built their whole number system around numbers like 60 and 360. These numbers have lots of divisors. They can be divided by 2, 3, 4, 5 and 6 without needing fractions. We have settled for the decimal system and have to put up with fractions such as $\frac{1}{3}$ being 0.333333....

The Babylonian system survives, not only in angle measurement, but also in time. We have the Babylonians to thank for having 60 minutes in one hour and 60 seconds in one minute.

The pyramid builders used knotted ropes. They divided a length of rope into 12 equal pieces, separated by knots and joined the ends together. Then stretching the loop into a triangle, so that the sides were 3, 4 and 5 the angle between the three and the four seemed to be a right angle, or at least close enough to a right angle. It was also discovered experimentally that a 5-12-13 triangle seemed to have the same property.

There was no way they could proved that the angle was exactly a right angle, but neither did they care. If it wasn’t exact it was close enough for all practical purposes. The Greek mathematician Euclid was the first to introduce the idea of proof into mathematics. The Greeks were great philosophers and logical argument was something that they developed. What better study to apply logic to than mathematics.

Euclid built up a whole theory of geometry, where proof built upon proof. His treatment of geometry was so good that his books were used as school textbooks, suitably translated and with explanatory footnotes, up until the early twentieth century.

Now you cannot prove anything from nothing. You have to start with some basic assumptions. Euclid considered these as self-evident but modern mathematicians do not regard them as such. Their attitude is that if you accept Euclid’s basic premises then all his theorems follow logically. But on the truth or otherwise of the basic assumptions they remain silent.

Euclid begins by talking about **points** and **lines** (circles come later). These are undefined concepts. We think of lines as being straight, though there is the question of what we mean by straight. You see that understanding geometry requires some form of geometric intuition that seems to be hard-wired in our brains.

It is possible to set up geometry so that a disembodied angel – an intelligent being with no concept of the special world – could understand at some level, but even then there would be underlying assumptions. The difference is that to us these assumptions would seem reasonable, based on our spatial intuition, while to the disembodied angel they would seem quite arbitrary.

Also undefined is the notion of a point **lying on** a line, or equivalently, a line **passing through** a point. A fundamental assumption of Euclid is that given any two distinct points there is exactly one line that passes through them. If A and B are two points then AB (or BA) denotes this unique line.

This seems reasonable enough, but remember that if a line is defined as the shortest distance between two points then on the surface of a sphere there are infinitely many lines joining the north and south pole. The lines of longitude all go from pole to pole and they all have the same length.

Alright, we are talking about a flat plane. But what exactly do we mean by flat? If we plot two points on a piece of paper it seems reasonable that there is only one straight line between them. But might there not be several lines very close together joining the points all with exactly the same length.

Two lines **intersect** at a point A if A lies on them both. Do any two distinct lines intersect in exactly one point? No, there is the phenomenon of parallel lines. Two lines are defined to be **parallel** if they have no point of intersection. Informally we can consider parallel lines as lines going in the same direction, or keeping the same distance apart, but no intersection is the true definition. We write the statement that AB is parallel to CD by:

AB || CD.

A very important assumption that Euclid used, though oddly enough he didn’t include it in his list of postulates, is that if you have any line and any point P that doesn’t lie on that line, there is exactly one line through P that is parallel to the given line.

If we experiment with lines on a piece of paper it seems to be true. But is it exactly true? Again there may be several possible lines so close together that our drawing instruments can’t resolve them. And the line we draw, if continued indefinitely would not be exactly parallel, because geometric constructions in practice can never be completely exact.

In the 19th century mathematicians questioned this assumption and built up alternative, or non-Euclidean, geometries. Then, in the twentieth century physicists discovered that space in the vicinity of the nucleus of an atom is curved. Luckily there were already these well-developed non-Euclidean geometries for them to take off the shelf and use.

The role of a mathematician is not to determine the truth of the basic assumptions of geometry. Their role is to develop the consequences of the premises. Non-Euclidean geometries are quite an advanced part of mathematics so we will only be discussing Euclidean geometry.

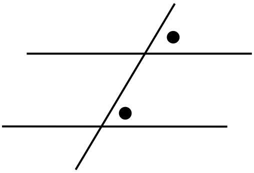
§4.2. Angles

An **angle** is a measure of rotation. If a line is rotated about a point that lies on it the angle between the original line and the rotated line is a measure of how far the line has been rotated. At this basic level the direction is assumed to be in the direction of the smallest rotation, so angles will lie between 0° and 180°. An **acute** angle is one that lies between 0° and 90° and an **obtuse** angle is one that lies between 90° and 180°.

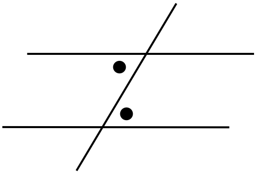
We denote the angle between the lines AB and AC as ∠BAC or ∠CAB. The special angle of 90° is called a **right angle** and 180° is called a **straight angle**. If the angle between two lines is 90° we say that the lines are **perpendicular**.

In some contexts we insist on the rotation going in a particular direction, in which case the angle lies between 0° and 360°. An angle that is above 180° is called a **reflex angle**.

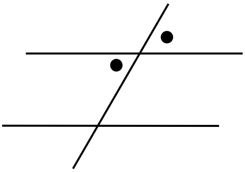
Suppose that you have two parallel lines and a transversal, that is a third line that is not parallel to the other two. It cuts them, and in so doing, eight angles are formed. Angles are said to be **corresponding**, **alternate** or **opposite** as follows.



CORRESPONDING



ALTERNATE



OPPOSITE

In each case we have used the same symbol, a dot, to represent the two angles. There is a convention in geometry that if we put the same symbol in two angles they are assumed to be equal. And Euclid assumes, what seems quite obvious to those of us who have spatial intuition, that corresponding angles are equal, as are alternate angles and opposite angles.

A moment’s thought will reveal that if we make any two of these assumptions the third one will follow. Even a disembodied angel will be able to see this – it requires no geometric intuition.

§4.3. Triangles

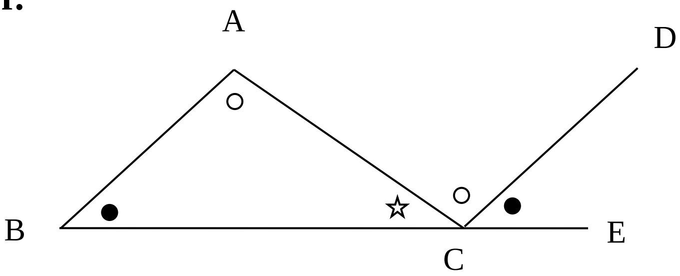
A **triangle** is a shape that is bounded by three straight sides. Enclosed between these sides are three angles. We usually describe the triangle by giving symbols to the **vertices** (corners). So **ΔABC** denotes the triangle whose vertices are A, B and C.

The **sides**, or edges, of the triangle are denote by giving the endpoints. So the sides of ΔABC are AB, AC and BC. The order of the symbols representing each side is irrelevant, so we could use BA instead of AB.

The three **angles of the triangle** ΔABC are ∠ABC, ∠BCA and ∠CAB. The symbol in the middle of the expression represents the **vertex of the angle**, that is, the vertex that is the centre of the rotation.

We can use the parallel postulate to prove that the angles of a triangle total 180°.

Theorem 1: The sum of the angles of a triangle total 180°.
Proof:

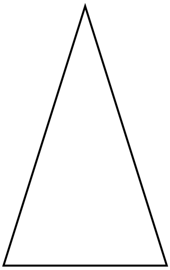


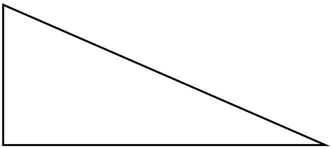
Draw the line CD, through C, parallel to BA.
∠ABC = ∠DCE (corresponding angles).
∠BAC = ∠ACD (alternate angles).
∴∠ABC + ∠BAC + ∠ACB
= ∠ACB + ∠ACD + ∠ DCE = 180°.

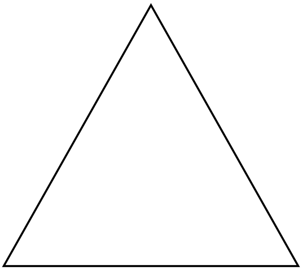
There are special names for certain special types of triangle. If two sides of a triangle are equal in length we call it an **isosceles** triangle. If all three sides have the same length the triangle is called an **equilateral** triangle.

The prefixes ‘isos-‘ and ‘equi-‘ both mean equal and ‘celes’ and ‘lateral’ both refer to sides. The word ‘isosceles’ seems to be one of the most difficult to spell. Over a period of years marking HSC exam papers my marking team collected all the variant spellings of the word. We found over 300 variants!

A **right-angled triangle** is one where one of the angles is 90°. The side opposite the right-angle is called the hypotenuse.

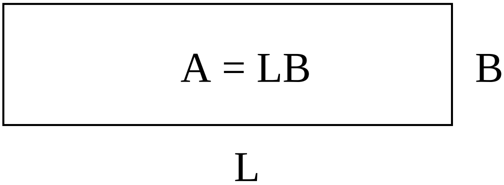






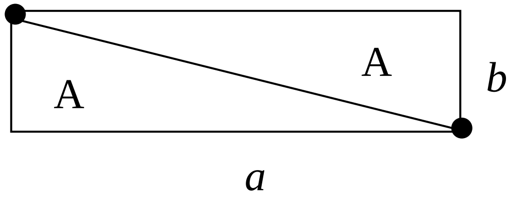
ISOSCELES RIGHT-ANGLED EQUILATERAL

We define the **area** of rectangles as “length times breadth”. The terms length and breadth refer to the lengths of two adjacent sides. Which one is called the length and which one the breadth is irrelevant.



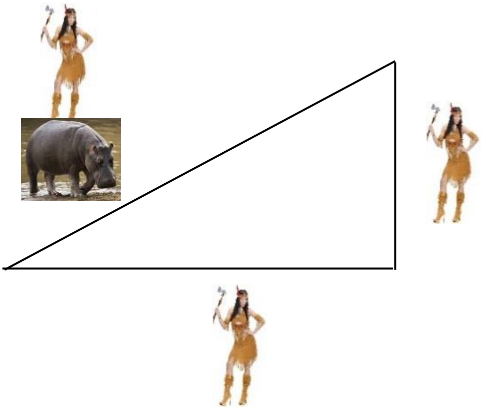
We can find areas of certain other shapes by using the principle that congruent shapes have the same area. Two shapes are **congruent** if you can move one (rotating and translating) so that it coincides with the other. Informally, congruent shapes have the same shape and size. This enables us to find the area of a right-angled triangle. We can cut a rectangle into two congruent right-angled triangles, so the area of each triangle is half that of the rectangle.

$$A = \tfrac{1}{2} ab$$



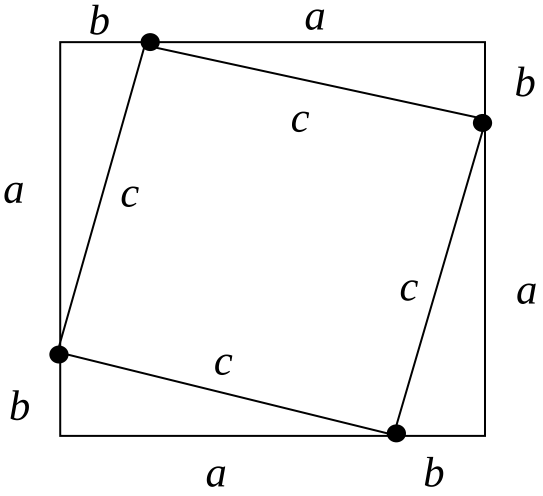
This gives us the formula that **the area of a right-angled triangle is half the product of the shorter two sides**.

A famous theorem, known as Pythagoras’ theorem, states that “the square on the hypotenuse is the sum of the squares of the other two sides”. This has been parodied as “the squaw on the hippopotamus is the sum of the squaws on the other two hides”!



Theorem 1 (Pythagoras): If the lengths of the sides of a right-angled triangle are a, b and c, where c is the length of the hypotenuse, then $c^2 = a^2 + b^2$.

Proof:



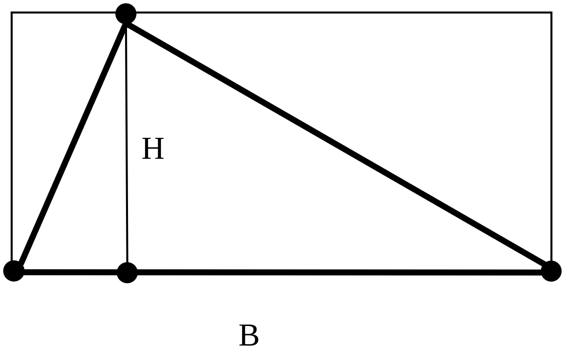
The area of the outer square is $(a + b)^2 = a^2 + b^2 + 2ab$. The area of the inner square is c^2. The area of each right-angled triangle is ½ ab, so together the four triangles make up an area of $2ab$.

$\therefore a^2 + b^2 + 2ab = c^2 + 2ab$ and so $c^2 = a^2 + b^2$.

This proof is just one of literally hundreds of different proofs for this famous theorem. There is a book that consists of nothing else but proof after proof of Pythagoras’ Theorem. In fact some years ago the *Guinness Book of Records* listed Pythagoras’ Theorem as the theorem in mathematics that has the most number of different proofs. They claimed there were over three hundred of them!

While on the subject of areas, how do we work out the area of a triangle that is not right-angled? We choose one side and call it the base (any side will do). Then we construct a rectangle around it as follows.

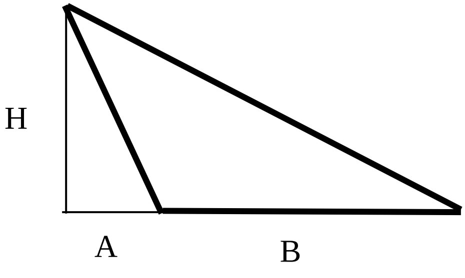
$$A = \tfrac{1}{2} HB$$



We divide the triangle into two right-angled triangles by drawing the line through the other vertex that is perpendicular to the base. The length of this line is called the **altitude**. Here we have called the altitude H. [Remember that we could take any side as the base so we actually have three altitudes.]

The surrounding rectangle is subdivided into four right-angled triangles – two pairs of congruent ones. The triangle is made up of one of each pair. So it is clear that the area of the triangle is one half that of the rectangle.

One has to be careful in proving geometrical facts by diagrams, because the diagram might not reflect all possible cases. If a triangle has an obtuse angle (greater than 90°) the above argument will not work. Such a triangle is called a **scalene triangle**. Instead of making the triangle the *sum* of two right-angled triangles we make it the *difference*.



The area of the scalene triangle is ½ H(A + B) – ½ HA = ½ HB. Again "half the base times the perpendicular height" works.

§4.4. Parallelograms

A **quadrilateral** is a shape bounded by four straight sides. Quadrilaterals include squares and rectangles. Another interesting type of quadrilateral is the parallelogram. It looks like a squashed rectangle. A **parallelogram** is a quadrilateral where opposite sides are equal. If these sides have equal length it is called a **rhombus**.

The area of a parallelogram is **base times perpendicular height** because a parallelogram can be divided into two congruent triangles, with equal sized bases and the same perpendicular height.

There are many properties of parallelograms that can be proved along the same lines as our earlier proof. We do not provide these proofs here, but just list these properties.

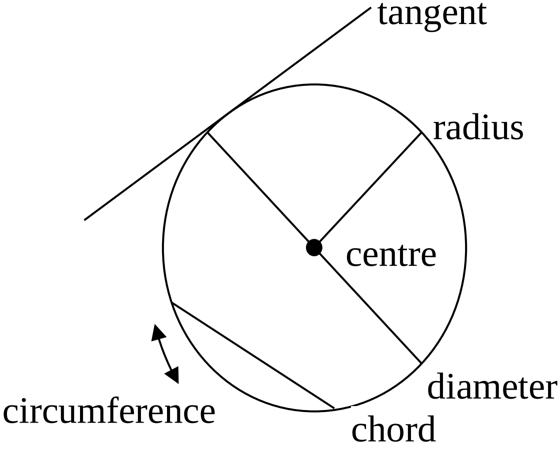
- opposite sides of a parallelogram are equal in length
- opposite angles of a parallelogram are equal
- the two diagonals of a parallelogram bisect each other
- the two diagonals of a rhombus are perpendicular to each other

§4.5. Circles

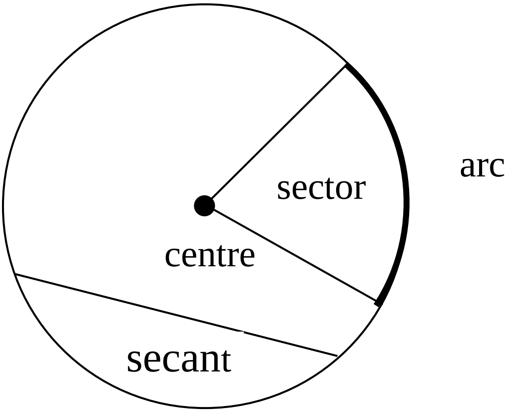
After considering the geometry of straight lines, and shapes bounded by straight line, Euclidean geometry moves on to circles. A **circle** is the **locus** of a point traced out by a point that moves so that its distance from some fixed point is constant, [Locus just means the path traced out by a moving point.] The fixed point is called the **centre** and any line from the centre to the circle is called a **radius**. Any line through the centre, from one side of the circle to the other, is called a **diameter**. The length of any radius is called *the* radius and the length of any diameter is called the diameter of the circle. Clearly the diameter is twice the radius.

The word ‘circle’ refers to both the outside curve, as well as to the whole region inside. When we talk about the **circumference** of the circle we mean the distance around the outside. When we talk about the area, we mean the whole region enclosed. The context will make our meaning clear. An **arc** is a portion of the circumference.

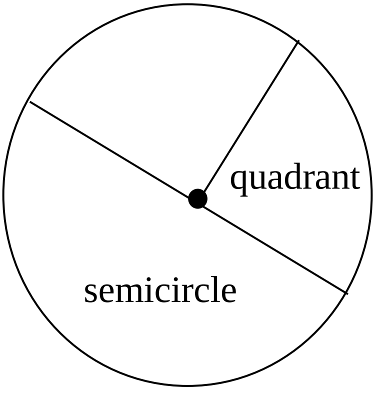
A **chord** of a circle is any line whose endpoints are on the circumference and an arc is any piece of the circumference. A **tangent** is any line that touches the circle in exactly one point.



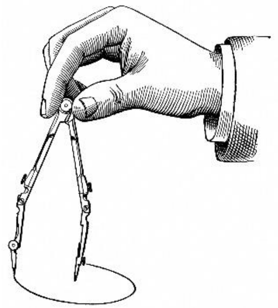
A **sector** of a circle is a region that is bounded on two sides by **radii** (plural of radius) and on the third side by an arc. It is like a slice of pie. A **secant** of a circle is a region bounded on one side by a chord and on the other by an arc.



A **semicircle** is a secant where the chord is a diameter, that is, it is exactly one half of the circle. A **quadrant** of a circle is a sector where the radii are at right angles, that is, it is exactly one quarter of the circle.



An instrument for drawing circles is called a **pair of compasses**. There are two hinged arms. At the tip of one arm is a sharp point that digs into the paper at the centre. The other arm has an attachment that holds a pencil. As the instrument rotates the tip of the pencil traces out the circle.



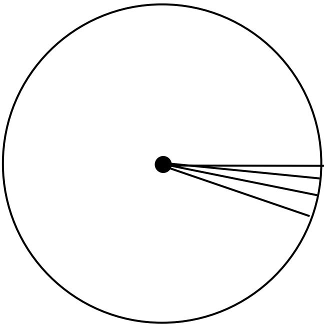
If we double the diameter we double the circumference. So the circumference will be a constant multiple of the diameter. If we draw some circles and measure the circumference (by stretching a piece of string around the outside) we find that this ratio is about 3. The circumference is about three times the diameter.

Gradually, better and better approximations were found. But there must be an exact value. This value is the number pi (π in symbols) that we talked about in Chapter 2.

So if r is the radius (half the diameter), and C is the circumference, we have $\mathbf{C = 2\pi r}$.

The area is a little more difficult. How do we measure the area? The area of any shape that can be cut up into triangles can be found by working out the sum of the areas of the triangles. But, no matter how many pieces we cut a circle into, some of the pieces will have a curved edge.

Suppose we take a circle of radius r and cut it up into 100 equal sized sectors.



Each of the 100 sectors will be approximately an isosceles triangle. The base will be approximately one hundredth of the circumference, that is, $\frac{2\pi r}{100}$. The perpendicular height will be the radius, r. So the area of each of the slender sectors will be $\left(\frac{1}{2}\right)\left(\frac{2\pi r}{100}\right) r$, that is, $\frac{\pi r^2}{100}$. So the area of the circle is approximately 100 times this, that is, πr^2.

A more careful analysis, using limits, shows that this is not just an approximation. It is exact. So the area of a circle with radius r is given by $\mathbf{A = \pi r^2}$.

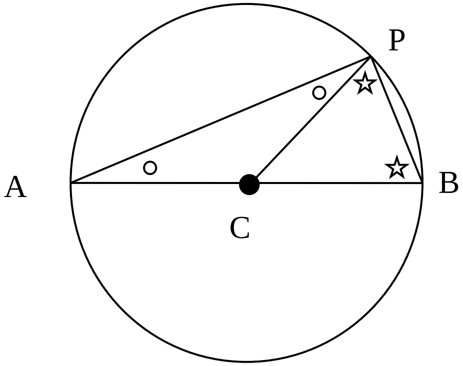
An important fact about circles is that **the angle in a semicircle is a right angle**. To prove this we use a fact about isosceles triangles, a fact that we won’t prove here. As well as having two equal sides, **an isosceles triangle has two equal angles**.

Theorem 2: If AB is a diameter of a circle and P lies on the circle then ∠APB = 90°.

Proof: Let C be the centre of the circle. Since ΔAPC is an isosceles triangle ∠PAC = ∠APC. These are marked as a small circle on the following diagram.

Since ΔPCB is an isosceles triangle ∠CPB = ∠PBC. These are marked with a small star.

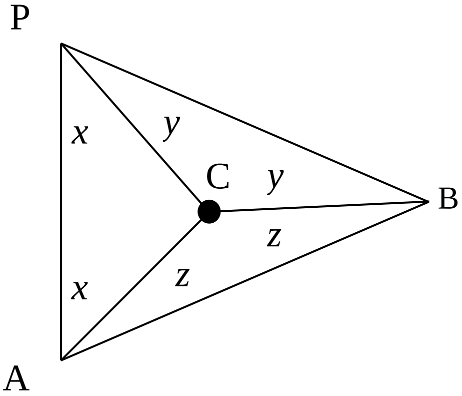
Now two circles plus two stars total 180° (the sum of the angles of ΔAPB), one circle plus a star must be 90° and so ∠APB is a right-angle.



This is a particular example of the theorem that the angle subtended by a chord at the centre of a circle is twice the angle subtended at any point on the circumference. The meaning of ‘subtend’ will be made clear by the following more careful statement.

Theorem 3: If A, B, P are three points on a circle, with centre C then ∠ACB = 2∠APB.

Proof: Note that ΔACP, ΔPCB and ΔACB are all isosceles. Let ∠CAP = *x*, ∠CBP = *y* and ∠CBA = *z*. Therefore ∠CPA = *x*, ∠CPB = *y* and ∠CAB = *z*.



Since the angles of ΔAPB total 180° we have:

$$2x + 2y + 2z = 180.$$

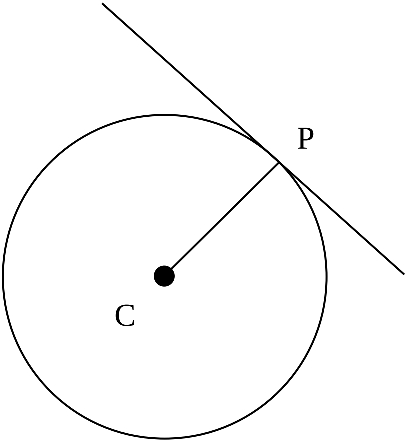
$\therefore\ x + y + z = 90.$

$\therefore\ \angle APB = x + y = 90 - z.$

Now the angles of ΔACB total 180° so
∠ACB = 180 – 2*z*, which is twice ∠APB.

There are numerous other theorems about circles that we won't prove here. But an important one is the following.

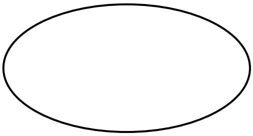
Theorem 4: The tangent at P, on a circle whose centre is C, is perpendicular to the radius CP.



Proof: The proof is more difficult than the previous ones and so is omitted.

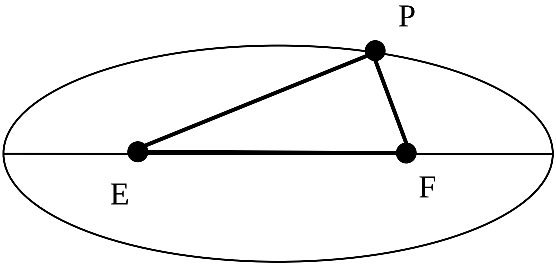
§4.6. Ellipses

Another shape that is of interest is the **ellipse**. This is like an elongated circle. Instead of one centre it has two special points, called **foci** (each one is a **focus**). It occurs in nature in so far as the orbits of the planets around the sun are actually ellipses. Perhaps you thought of them as circles, but they are not quite true circles.



Suppose you take two points E and F and have a third point P that moves so that the sum of its distances from E and F is a constant. The locus of P is the ellipse and E and F are the foci.

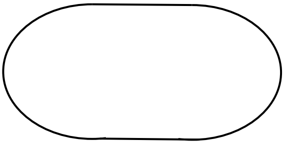
You can draw an ellipse by putting two pins into your paper at E and F (sticking into a backing board so that they stand up firmly.



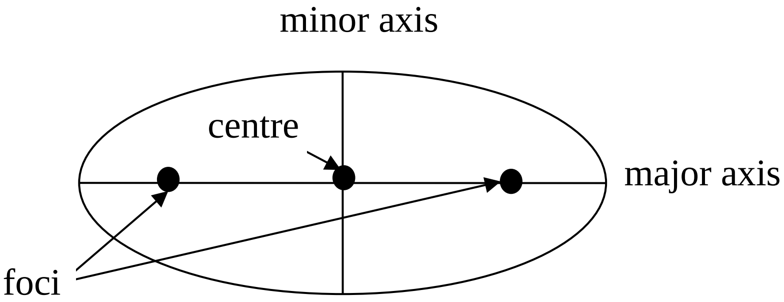
Then you take a string and make a loop by tying the ends together. You put this over the pins and pull it taut by putting a pencil into the loop to make a triangle ∆PEF. Then you move the pencil around the loop, making sure that the string is pulled tight at all times.

In the nineteenth century ellipses were discussed in most drawing manuals because it is a shape that is quite important in art. We see circles all around us but rarely do we see a circle straight on – usually they are seen at an angle. A circle, seen in perspective, appears as an ellipse and we need to draw them as such. We see them as circles because the brain adjusts for the perspective view but they are ellipses on the page.

These days art is more free-form but perspective was taken very seriously in those days. So a person learning to draw needed to learn how to draw an ellipse. Now an ellipse is not a rectangle with a couple of circles added on. That’s an oval.



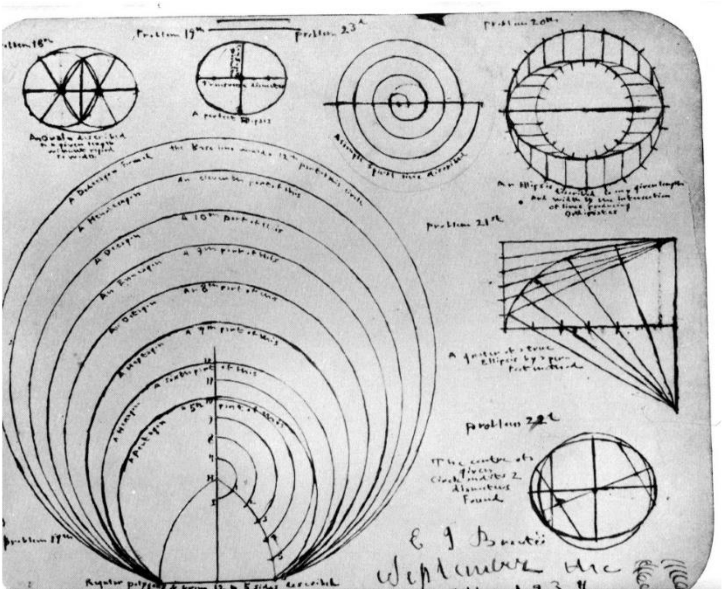
Nor is an ellipse egg-shaped. Eggs are fatter at one end and narrower at the other. An ellipse is symmetrical in both directions. The lines of symmetry are called the **major axis**, going through the foci, and the **minor axis**. Where these intersect is called the **centre** of the ellipse.



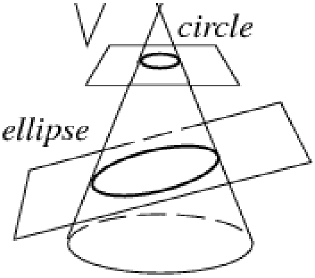
Emily Brontë, the author of *Wuthering Heights*, was the sister of Charlotte Brontë who wrote *Jane Eyre* and Ann Brontë who wrote *The Tenant of Wildfell Hall*.

Emily and Charlotte were very much interested in art, and at one stage in her youth Charlotte Brontë saw herself as making a living as an artist. Emily took the technical side of her drawing very seriously. Not only did she possess a set of geometric instruments, recently sold at auction, but she worked through drawing exercises from the fifth edition of *An Introduction to Perspective, Practical Geometry, Drawing and Painting* by Charles Hayter. The Brontë Parsonage museum has a manuscript of Emily's where she carries out some exercises on drawing ellipses.

I have examined the manuscript and held it up to the light. There were two tiny pinholes at the foci in Problem 19 showing that she had constructed that ellipse using string and pins. In other problems she constructed ellipses using other methods.

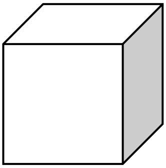


The ellipse is one of several shapes known as conic sections. If you take a cone and cut it by a plane perpendicular to the axis you get a circle. But if you cut it by a plane at a certain angle you get an ellipse. Other conic sections are parabolas and hyperbolas.

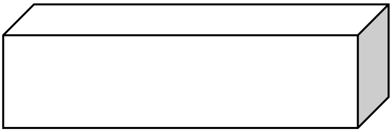


§4.7. Solid Geometry

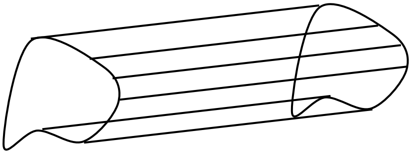
So far we have been talking about two-dimensional shapes only. The basic three-dimensional solid is the cube. It consists of 8 vertices, or corners, 6 faces and 12 edges. The volume V of a cube whose sides are each *x* units is given by $\mathbf{V = x^3}$.



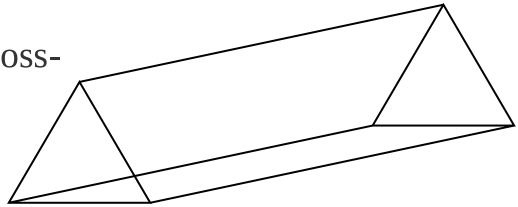
A box shape, where length, breadth and height are not necessarily all the same is called a **rectangular prism**. The volume of a rectangular prism is given by **V = LBH** where L is the length, B is the breadth and H is the height.



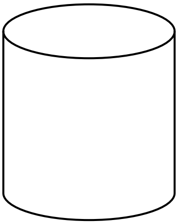
A **prism** in general is any plane shape that has been extruded in a third direction. The cross-sections by planes parallel to the ends have the same shape.



A **triangular prism** is a prism where the cross-sections are triangles. The most common example is where the cross sections are equilateral triangles (a Toblerone box.)

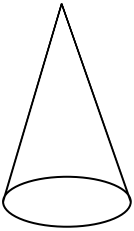


A circular prism is more commonly called a **cylinder**.

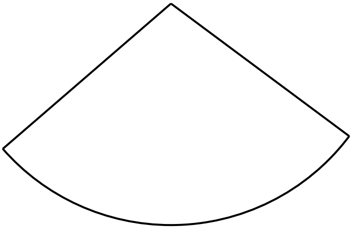


The formula for the volume of any prism is $\mathbf{V = A}\boldsymbol{h}$ where A is the area of each end and h is the height. The height is the distance between the ends. So the volume of a cylinder is $\mathbf{V} = \boldsymbol{\pi r^2 h}$

A **cone** is a solid that has a circular base but which tapers to a point. The volume of a cone is $\mathbf{V} = \boldsymbol{\frac{1}{3}\pi r^2 h}$ where r is the radius of the base and h is the height, that this the distance between the centre of the base and the tip of the cone. It is probably reasonable to expect some fraction times $\pi r^2 h$, but why it is one third must be left unsaid here.



Now if ever you have to make a cone from cardboard you take a sector of a circle and roll it up, joining the two radii. The arc of the sector becomes the circumference of the base. If you want a base you need to cut out a circle of the right size and join it on.



Example 1: Make a wizard's hat to fit a head 62.8 cm in circumference and with a vertical height of 40cm. Attach a brim 15cm wide. (Use the approximation 3.14 for π.)

Solution: There will be two pieces.

Hat: First we need to work out the radius, r, of the head. The circumference is 62.8 so $62.8 = 2\pi r \approx 6.28r$ so $r = 10$.

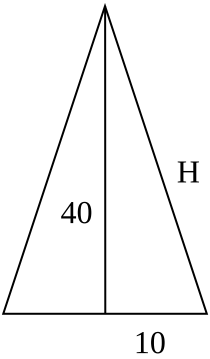
Now we need to work out the slant height H. This is the distance of a point on the circumference of the brim to the vertex of the hat.

If we take a vertical cross section of the hat through the *centre it* will be an isosceles triangle with H as the hypotenuse. The other two sides will be 10 and 40.

By Pythagoras' Theorem $H^2 = 10^2 + 40^2$

$= 100 + 1600 = 1700.$

$\therefore$ H = $\sqrt{1700} \approx 41.2$.



This will be the radius of the sector that we will roll up. Now we need to find the angle. Suppose we denote this angle by θ.

The circumference of the full circle is 2π times 41.2 which is approximately 259 cm. This corresponds to 360°.

$\therefore$ 1cm will correspond to $\frac{360}{259}$ degrees.

$\therefore$ 62.8cm will correspond to $\frac{360}{259} \times 62.8 \approx 87.3$ degrees.

No pair of compasses will draw an arc with a radius as big as 41.2 cm so we take a piece of string about 50cm long, tie a pencil at one end and anchor the other end on a drawing pin in the cardboard so that the distance between the tip of the pencil and the drawing pin is 41.2cm. Now draw an arc a bit more about 90 degrees. Draw a radius to one end of the arc. Measure an angle of 87 degrees and draw the other radius so that the angle of the sector is as close as you can get to 87 degrees. Now cut out the hat and roll it up. Sticky tape the two radii together making sure there is no overlap.

Brim: Draw a circle with radius 10 cm and a larger circle, with the same centre, with radius 25 cm. (The extra 15cm is the width of the brim.) Now cut out the inner circle and sticky tape the brim to the hat.

Example 2: We need to make 2 of these hats and the parts of the hats that are visible (the outside of the cone and both sides of the rim) when they're worn need to be painted black. Each tin of black paint says that there's enough to cover 0.3 square metres. How many tins should we buy?

Solution: The radius of the cone that will be rolled up to make the cone is 41.2 and the angle at the vertex is 87.3 degrees. The area of the total circle would be $41.2^2\pi$ = 5333 square centimetres. Since $\frac{87}{360} \times 5333 = 1289$, the cone will have a surface area of about 1289 square centimetres.

Then there is the rim. The external radius of the rim is 25 cm and the internal radius is 10 cm. So the surface area of each side of the rim is $25^2\pi - 10^2\pi = 525\pi = 1649$ square centimetres. So the total area to be painted for each hat is $1289 + 2 \times 1649 = 4587$ square centimetres. This is a little bit less than ½ square metre. So in all we'd need a smidgeon less than 1 tin. Perhaps we should buy one tin and hope that this will be just enough. Or perhaps we should buy 2 tins just to be on the safe side.

This exercise demonstrates a fundamental fact about mathematics. While it's very precise it doesn't, by itself, offer definitive answers to real world problems. It's a valuable tool, but in the end it is *we* who have to make the final decision, taking into account other factors, many of which are not susceptible to mathematical analysis.

For example, an aeronautical engineer might use mathematics to decide that a certain safety feature would reduce the possibility of a crash by 0.0001% per flight. He might choose to decide against it if would double the fares. Only human judgement can settle that. And although everyone will agree on the mathematics, different decisions on the practical outcomes will be possible.

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